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# On the relativistic dynamics of spinning matter in space–time with curvature and torsion

A V Minkevich and F Karakura

Department of Physics, Byelorussian State University, Minsk, USSR

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**Abstract.** A variational formalism is developed for conservative systems with internal degrees of freedom in a space–time with curvature and torsion. Lagrange's translational and rotational equations of motion are obtained as well as expressions for the energy-momentum and spin angular momentum tensors; Noether's identities are derived. Some properties of the developed formalism are studied in greater detail.

## 1. Introduction

At present, the model of so-called spinning matter (see Weyssenhoff and Raabe 1947, Halbwachs 1960, and also Maugin and Eringen 1972, Maugin 1974, 1978, Zhelnorovich 1980) is widely applied for the classical description of media possessing not only energy and momentum but also internal angular momentum. In a series of works (see the review of Hehl *et al* 1976), this model has been used in connection with the study of the influence of spin angular momentum and torsion of space–time on the structure of cosmological models in the frame of the Einstein–Cartan theory of gravitation. But it is well known that this model, for the time being, has no satisfactory variational formalism associated with it.

In this work a variational formalism is given for the relativistic dynamics of a conservative medium with internal degrees of freedom. The following developments are based on results obtained by Minkevich and Sokolski (1971, 1975), where a variational formalism was developed for media without internal degrees of freedom as well as for particles with momenta. According to the gauge approach in the theory of gravitation, the space–time continuum possesses both a curvature and a torsion. The medium is described by means of the following parameters: invariant densities of the conservative charges $q_l$ (for example: mass, electric charge, entropy) and the four-velocity of the medium particle $u^\mu$ satisfying the following conditions:

$$g_{\mu\nu}u^\mu u^\nu = -c^2, \tag{1}$$

$$\tilde{\nabla}_\mu(q_l u^\mu) = \overset{*}{\nabla}_\mu(q_l u^\mu) = 0 \qquad (\overset{*}{\nabla}_\mu = \nabla_\mu + 2S_{\mu\lambda}^{\cdot\cdot\lambda}), \tag{2}$$

where the symbols $\tilde{\nabla}_\mu$ and $\nabla_\mu$ denote covariant derivatives, calculated by means of the Christoffel symbols $\{{}^{\lambda}_{\mu\nu}\}$ and the total connection $\Gamma^\lambda_{\mu\nu}$, respectively. We have

$$\Gamma^\lambda_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\} + S_{\mu\nu}^{\cdot\cdot\lambda} + S^{\lambda}_{\cdot\mu\nu} + S^{\lambda}_{\cdot\nu\mu},$$

where $S_{\mu\nu}^{\cdot\cdot\lambda} = \frac{1}{2}(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) \equiv \Gamma^\lambda_{[\mu\nu]}$ is the torsion tensor; the signature of the metric $g_{\mu\nu}$ equals +2 and $c$ is the velocity of light. For the description of the internal degrees of freedom a triad of four-vectors $\underset{i}{l}^{\mu}$ ($i$ is the number of the vector) is connected with each particle of the medium. These vectors are spatial and orthonormal in the system of the particle mass rest (Maugin and Eringen 1972, Minkevich and Sokolski 1975). That is,

$$g_{\mu\nu}u^{\mu}\underset{i}{l}^{\nu}=0, \qquad g_{\mu\nu}\underset{i}{l}^{\mu}\underset{j}{l}^{\nu}=\delta_{ij}. \tag{3}$$

In connection with the choice of the above parameters in the frame of the studied model it is possible to consider spinning matter, and also media possessing both momenta connected with rotation of the particles and 'innated' momenta of various origins.

## 2. The motion equations

While describing the medium dynamics in the frame of the variational principle, the relations (1)–(3) play the role of constraints. To obtain the equations of translational motion, we shall vary the world lines of particles; the variation of other quantities in accordance with the conditions (1)–(3) is defined by variations in the displacement. It is possible to satisfy identically the conditions (1) and (2) by means of the following relations used by Fock (1961) for the four-velocity and the invariant charge densities:

$$u^{\alpha}=\frac{c\,\partial f^{\alpha}/\partial a_0}{[-g_{\mu\nu}(\partial f^{\mu}/\partial a_0)\partial f^{\nu}/\partial a_0]^{1/2}}, \tag{4}$$

$$q_l\sqrt{-g}\mathcal{J}=F_l(a_1,a_2,a_3)[-g_{\mu\nu}(\partial f^{\mu}/\partial a_0)\partial f^{\nu}/\partial a_0]^{1/2}, \tag{5}$$

where $a_0$, $a_1$, $a_2$, $a_3$ are Lagrange's coordinates of a particle, connected with Euler's coordinates $x^{\mu}=f^{\mu}(a_0, a_1, a_2, a_3)$, $\mathcal{J}=D(x^0, x^1, x^2, x^3)/D(a_0, a_1, a_2, a_3)$ is the Jacobian of the motion and the functions $F_l(a_1, a_2, a_3)$ depend on the distribution of densities $q_l$. The variation in displacement of the particle is

$$\begin{aligned} &x^{\mu}(a_0,a_1,a_2,a_3)\rightarrow x^{\mu}(a_0,a_1,a_2,a_3)+\delta x^{\mu},\\ &\delta x^{\mu}=\delta f^{\mu}(a_0,a_1,a_2,a_3)=\xi^{\mu}(x^0,x^1,x^2,x^3), \end{aligned} \tag{6}$$

which, on account of (4) and (5), reduces to the following variations:

$$\delta_F u^{\mu}=-\xi^{\lambda}\tilde{\nabla}_{\lambda}u^{\mu}+n^{\mu}_{\lambda}u^{\nu}\tilde{\nabla}_{\nu}\xi^{\lambda}, \tag{7}$$

$$\delta_F q_l=-\xi^{\lambda}\tilde{\nabla}_{\lambda}q_l-q_l n^{\nu}_{\lambda}\tilde{\nabla}_{\nu}\xi^{\lambda}, \tag{8}$$

where $n^{\mu}_{\nu}=\delta^{\mu}_{\nu}+c^{-2}u^{\mu}u_{\nu}$. Then the variations of triad fields by virtue of (3) and (7) have the form

$$\delta^{(1)}_F\underset{i}{l}^{\mu}=-\xi^{\lambda}\tilde{\nabla}_{\lambda}\underset{i}{l}^{\mu}+u^{\mu}\underset{i}{l}_{\lambda}u^{\nu}\tilde{\nabla}_{\nu}\xi^{\lambda}. \tag{9}$$

In a space with torsion the variations $\delta_F u^{\mu}$ and $\delta_F q_l$ can be represented in the form

$$\delta_F u^{\mu}=-\xi^{\lambda}\nabla_{\lambda}u^{\mu}+n^{\mu}_{\lambda}u^{\nu}\xi^{\lambda}_{\cdot\nu}, \tag{10}$$

$$\delta_F q_l=-\xi^{\lambda}\nabla_{\lambda}q_l-q_l n^{\nu}_{\lambda}\xi^{\lambda\nu}_{\cdot\cdot}, \tag{11}$$

where $\xi^{\lambda}_{\cdot\nu}=\nabla_{\nu}\xi^{\lambda}+2S_{\rho\nu}^{\cdot\cdot\lambda}\xi^{\rho}$.

Note that the expression (10) for $\delta_F u^\mu$ is in full agreement with the method of covariant variation in the relativistic dynamics of particles (Minkevich and Fedorov 1968)†. According to the definition, the covariant variation $\delta_K$ is the difference between the varied quantity at the point $x^\mu + \delta x^\mu$ and the non-varied quantity transferred parallelly from the point $x^\mu$ to the point $x^\mu + \delta x^\mu$. The variation (10) represents the following difference:

$$\delta_F u^\mu = \delta_K u^\mu - \xi^\lambda \nabla_\lambda u^\mu.$$

The application of the formula for the covariant variation $\delta_K \underset{i}{l}^\mu$ (Minkevich and Sokolski 1975) gives the expression

$$\delta_F^{(2)} \underset{i}{l}^\mu = \delta_K \underset{i}{l}^\mu - \xi^\lambda \nabla_\lambda \underset{i}{l}^\mu = -\xi^\lambda \nabla_\lambda \underset{i}{l}^\mu + c^{-2} u^\mu \underset{i}{l}_\lambda u^\nu \xi^\lambda_{\cdot;\nu}. \tag{12}$$

The variations (9) and (12) differ by the quantity

$$(\delta_F^{(2)} - \delta_F^{(1)}) \underset{i}{l}^\mu = n^\mu_\nu \underset{i}{l}_\lambda (S^{\lambda\nu}_{\cdot\cdot\rho} - 2S_\rho^{\cdot[\lambda\nu]})\xi^\rho. \tag{13}$$

However, this difference is not essential since it is a particular case of the variation $\delta_r \underset{i}{l}^\mu$ corresponding to the triad rotation:

$$\delta_r \underset{i}{l}^\mu = n^\mu_\beta \underset{i}{l}_\alpha \varepsilon^{\alpha\beta}, \tag{14}$$

where $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ are six arbitrary infinitesimal functions. Note that the variation (14), by means of the formula $n^\mu_\beta = \underset{i}{l}^\mu \underset{i}{l}_\beta$, may be represented in the form

$$\delta_r \underset{i}{l}^\mu = \underset{ij}{\varepsilon} \underset{j}{l}^\mu, \tag{15}$$

where $\underset{ij}{\varepsilon} = -\underset{ji}{\varepsilon}$ are three independent infinitesimal parameters defining the rotation of the vectors $\underset{i}{l}^\mu$. The variation $\delta_F$ is the difference at the same point of Euler's system of coordinates and for the functions independent of $q_l$, $u^\mu$ and $\underset{i}{l}^\mu$ vanishes identically. Note that the operation $\delta_F$ commutes both with the particular and covariant differentiation.

We consider the system composed of a continuous medium with internal degrees of freedom and non-geometric tensor fields $Q_a$, interacting with one another and with the gravitational field described by the metric tensor $g_{\mu\nu}$ and the torsion tensor $S_{\mu\nu}^{\cdot\cdot\lambda}$ ‡.) Taking into account the principle of minimal coupling we represent the action integral in the form

$$I = c^{-1} \int_{(\Omega_4)} L(A_B, \nabla_\mu A_B, g_{\mu\nu}) \sqrt{-g}\, d^4x, \tag{16}$$

where

$$A_B = \{q_l, u^\mu, \underset{i}{l}^\mu, Q_a\},$$

$$\nabla_\mu A_B = \partial_\mu A_B + \Gamma^\lambda_{\mu\nu} A_B|^\nu_{\cdot\lambda}, \qquad A_B|^\nu_{\cdot\lambda} = A_C a_B{}^C|^\nu_{\cdot\lambda}.$$

† Note that the absolute derivative along the trajectory of the particle when passing to a continuous distribution of matter, corresponds to the operator $u^\mu \nabla_\mu$.

‡ For the consideration of spinor fields it is necessary to introduce orthonormal tetrad fields $h^i{}_\mu$ which are connected with the metric tensor as $g_{\mu\nu} = \eta_{ik} h^i{}_\mu h^k{}_\nu$ ($\eta_{ik} = \mathrm{diag}(-1, 1, 1, 1)$). In the framework of the gauge theory of gravity the gravitational field is described then by means of the tetrads and the rotating Ricci coefficients.

The coefficients $a_B{}^C|^{\nu}_{\cdot\lambda}$ are expressed by means of Kronecker symbols and their explicit form depends on the order of the tensors $A_B$. Using (7)–(9) and varying the motion of the medium, applying Gauss's theorem in covariant form and supposing that, at the boundary of the four-domain of integration, $\xi^\mu = 0$ and $\xi^\mu_{,\nu} = 0$, in conformity with Hamilton's principle $\delta_F \mathcal{I} = 0$ we will find the equations of translational motion of the medium as

$$\tilde{\nabla}_\lambda\left(\frac{\delta L}{\delta u^\mu} n^\mu_\nu u^\lambda + \frac{1}{c^2}\frac{\delta L}{\delta \underset{i}{l}^\mu} u^\mu \underset{i}{l}_\nu u^\lambda - \frac{\delta L}{\delta q_l} q_l n^\lambda_\nu\right) + \frac{\delta L}{\delta u^\mu}\tilde{\nabla}_\nu u^\mu + \frac{\delta L}{\delta \underset{i}{l}^\mu}\tilde{\nabla}_\nu \underset{i}{l}^\mu + \frac{\delta L}{\delta q_l}\tilde{\nabla}_\nu q_l = 0, \tag{17}$$

where $\delta/\delta A_B = \partial/\partial A_B - \overset{*}{\nabla}_\mu \partial/\partial(\nabla_\mu A_B)$.

The use of the variations (10)–(12) instead of formulae (7)–(9) will give

$$(\overset{*}{\nabla}_\lambda \delta^\alpha_\nu - 2S_{\nu\lambda}^{\cdot\cdot\alpha})\left(\frac{\delta L}{\delta u^\mu} n^\mu_\alpha u^\lambda + \frac{1}{c^2}\frac{\delta L}{\delta \underset{i}{l}^\mu} u^\mu \underset{i}{l}_\alpha u^\lambda - \frac{\delta L}{\delta q_l} q_l n^\lambda_\alpha\right)$$
$$+ \frac{\delta L}{\delta u^\mu}\nabla_\nu u^\mu + \frac{\delta L}{\delta \underset{i}{l}^\mu}\nabla_\nu \underset{i}{l}^\mu + \frac{\delta L}{\delta q_l}\nabla_\nu q_l = 0. \tag{18}$$

Similarly, we will find the equations of rotational motion by varying the action integral (16) and using the variation (14):

$$(\delta L/\delta \underset{i}{l}^\mu)\underset{i}{l}_{[\alpha} n^\mu_{\beta]} = 0. \tag{19}$$

The application of the variation (15) instead of (14) gives the equations

$$(\delta L/\delta l^\mu_{[i})\, l^\mu_{j]} = 0. \tag{20}$$

Due to (3), equations (19) and (20) are equivalent. Equations (17) and (18) are equivalent by virtue of the equations of rotational motion (19).

## 3. Energy–momentum and spin angular momentum tensors

From the invariance of the Lagrangian $L$ there follows the relation

$$\underset{\mathrm{L}}{\delta} L = \zeta^\nu \nabla_\nu L, \tag{21}$$

where $\underset{\mathrm{L}}{\delta}$ is Lie's differential corresponding to the infinitesimal coordinate transformation $x^\mu \to x^\mu + \zeta^\mu$. Using the formulae

$$\underset{\mathrm{L}}{\delta} A_B = \zeta^\lambda \nabla_\lambda A_B - A_B|^{\nu}_{\cdot\lambda}\zeta^\lambda_{\cdot\nu}, \qquad \underset{\mathrm{L}}{\delta} g_{\mu\nu} = \zeta_{\mu\nu} + \zeta_{\nu\mu} \equiv 2\zeta_{(\mu\nu)},$$

where $\zeta_{\lambda\nu} = \nabla_\nu \zeta_\lambda + 2S_{\rho\nu\lambda}\zeta^\rho$, we transform the relation (21) to the form

$$\overset{*}{\nabla}_\lambda\left(L\zeta^\lambda - \frac{\partial L}{\partial(\nabla_\lambda A_B)}\underset{\mathrm{L}}{\delta} A_B\right) = \frac{\partial L}{\partial A_B}\underset{\mathrm{L}}{\delta} A_B + \tfrac{1}{2}\left(Lg^{\mu\nu} + 2\frac{\partial L}{\partial g_{\mu\nu}}\right)\underset{\mathrm{L}}{\delta} g_{\mu\nu} + \tfrac{1}{2}\sigma^{\lambda\cdot\mu}_{\cdot\nu}\underset{\mathrm{L}}{\delta}\Gamma^\nu_{\mu\lambda}, \tag{22}$$

where $\sigma^{\lambda\cdot\mu}_{\cdot\nu} = 2\,(\partial L/\partial(\nabla_\mu A_B))A_B|^{\lambda}_{\cdot\nu}$.

Taking into account the field equations $\delta L/\delta Q_a = 0$ and the equations of translational motion (18), we find

$$\frac{\delta L}{\delta A_B}\underset{\mathrm{L}}{\delta} A_B = \overset{*}{\nabla}_\nu\left[\zeta^\lambda\left(\frac{\delta L}{\delta q_l} q_l n_\lambda^\nu - \frac{\delta L}{\delta u^\mu} n_\lambda^\mu u^\nu - \frac{1}{c^2}\frac{\delta L}{\delta \underset{i}{l}^\mu} u^\mu \underset{i}{l}_\lambda u^\nu\right)\right]$$
$$-\zeta^\lambda_{\cdot\nu}\left[\frac{\delta L}{\delta q_l} q_l n_\lambda^\nu + \frac{\delta L}{\delta \underset{i}{l}^\lambda} \underset{i}{l}^\nu - \frac{1}{c^2}\left(\frac{\delta L}{\delta u^\mu} u_\lambda + \frac{\delta L}{\delta \underset{i}{l}^\mu} \underset{i}{l}_\lambda\right) u^\mu u^\nu\right]. \tag{23}$$

On taking into account the equations of rotational motions (19) and formula (23), the relation (22) may be transformed to the form

$$\overset{*}{\nabla}_\nu(2\zeta^\lambda t_\lambda^{\cdot\nu} + \sigma^{\rho\lambda\nu}\zeta_{\lambda\rho}) = \tau^{\mu\nu}\underset{\mathrm{L}}{\delta} g_{\mu\nu} + \sigma^{\lambda\cdot\mu}_{\cdot\cdot\nu}\underset{\mathrm{L}}{\delta}\Gamma^\nu_{\mu\lambda}, \tag{24}$$

where the following tensors are defined:

$$t_\lambda^{\cdot\nu} = L\delta_\lambda^\nu - \frac{\partial L}{\partial(\nabla_\nu A_B)}\nabla_\lambda A_B - \frac{\delta L}{\delta q_l} q_l n_\lambda^\nu + \frac{\delta L}{\delta u^\mu} n_\lambda^\mu u^\nu + \frac{1}{c^2}\frac{\delta L}{\delta \underset{i}{l}^\mu} u^\mu \underset{i}{l}_\lambda u^\nu, \tag{25}$$

$$\tau^{\mu\nu} = Lg^{\mu\nu} + 2\frac{\partial L}{\partial g_{\mu\nu}} - \frac{\delta L}{\delta q_l} q_l n^{\mu\nu} + \frac{1}{c^2}\frac{\delta L}{\delta u^\lambda} u^\lambda u^\mu u^\nu - \frac{\delta L}{\delta \underset{i}{l}^\lambda}\underset{i}{l}^{(\mu}\left(g^{\nu)\lambda} - \frac{1}{c^2} u^{\nu)} u^\lambda\right). \tag{26}$$

By using the formula $\underset{\mathrm{L}}{\delta}\Gamma^\lambda_{\mu\nu} = \nabla_\mu \zeta^\lambda_{\cdot\nu} + \zeta^\rho R^{\cdot\lambda}_{\cdot\nu\rho\mu}$, where $R^{\rho}_{\cdot\lambda\mu\nu} = 2\partial_{[\mu}\Gamma^\rho_{\nu]\lambda} + 2\Gamma^\rho_{[\mu|\alpha|}\Gamma^\alpha_{\nu]\lambda}$ is the curvature tensor, it is easy to obtain Noether's relations from (24):

$$(\overset{*}{\nabla}\nu\delta_\rho^\lambda - 2S_{\rho\nu}^{\cdot\cdot\lambda})t_\lambda^{\cdot\nu} = \tfrac{1}{2}\sigma^{\nu\lambda\mu}R_{\lambda\nu\rho\mu}, \tag{27}$$

$$t^{\lambda\nu} = \tau^{\lambda\nu} - \tfrac{1}{2}\overset{*}{\nabla}_\mu\sigma^{\nu\lambda\mu}. \tag{28}$$

Due to the symmetry of the tensor $\tau^{\lambda\nu}$, from (28) it follows that

$$\overset{*}{\nabla}_\mu\sigma^{[\nu\lambda]\mu} = 2t^{[\nu\lambda]}. \tag{29}$$

The tensors $t_\lambda^{\cdot\nu}$ and $\sigma^{[\nu\lambda]\mu}$ are, respectively, the canonical energy–momentum and spin angular momentum tensors. The form of the tensors essentially depends on the choice of the determining parameters of media and constraints (1)–(3) (cf Maugin and Eringen 1972, Zhelnorovich 1980). Equations (27) and (29) give the laws of their variation.

Varying the action integral (16) with respect to the metric, we can obtain the metric energy–momentum tensor $\theta^{\mu\nu}$. It is necessary to take into account not only the obvious dependence of the Lagrangian $L$ on the metric tensor but also that due to (1)–(3). The variations in the four-velocity $u^\mu$, the density of charges $q_l$ and the triad $\underset{i}{l}^\mu$ with respect to the metric differ from zero and are given by (Fock 1961, Minkevich and Sokolski 1975)

$$\delta_g u^\mu = (2c^2)^{-1}u^\mu u^\alpha u^\beta \delta g_{\alpha\beta}, \qquad \delta_g q_l = -\tfrac{1}{2}q_l n^{\alpha\beta}\delta g_{\alpha\beta},$$
$$\delta_g \underset{i}{l}^\mu = \tfrac{1}{2}(c^{-2}u^\mu u^\alpha - g^{\mu\alpha})\underset{i}{l}^\beta \delta g_{\alpha\beta}. \tag{30}$$

Using (30), as also the formula

$$\delta_g\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}g^{\lambda\alpha}[\nabla_\nu(\delta g_{\alpha\mu}) + \nabla_\mu(\delta g_{\nu\alpha}) - \nabla_\alpha(\delta g_{\mu\nu})], \tag{31}$$

we obtain the following expression for the tensor $\theta^{\mu\nu}$:

$$\theta^{\mu\nu}=Lg^{\mu\nu}+2\frac{\partial L}{\partial g_{\mu\nu}}-\frac{\delta L}{\delta q_l}q_l n^{\mu\nu}+\frac{1}{c^2}\frac{\delta L}{\delta u^\lambda}u^\lambda u^\mu u^\nu$$
$$+\frac{\delta L}{\delta \underset{i}{l}^\lambda}\left(\frac{1}{c^2}u^\lambda u^{(\mu}-g^{\lambda(\mu}\right)\underset{i}{l}^{\nu)}-\tfrac{1}{2}\overset{*}{\nabla}_\lambda(\sigma^{\lambda(\mu\nu)}+\sigma^{(\mu\nu)\lambda}-\sigma^{(\mu|\lambda|\nu)}). \tag{32}$$

Comparing formulae (26) and (32), it is easy from (28) to obtain the relation between the metric and canonical energy–momentum tensors. This has the usual form:

$$\theta^{\mu\nu}=t^{\mu\nu}+\tfrac{1}{2}\overset{*}{\nabla}_\lambda(\sigma^{[\mu\lambda]\nu}+\sigma^{[\nu\lambda]\mu}-\sigma^{[\mu\nu]\lambda}). \tag{33}$$

## 4. Variational formalism for Lagrangians depending on the angular velocity tensor

Let us consider the case when the Lagrangian $L$ depends on the derivatives of the four-velocity $u^\mu$ and the vectors $\underset{i}{l}^\mu$ through the tensor

$$\Omega^{\mu\nu}=\underset{i}{l}^\mu u^\lambda\nabla_\lambda \underset{i}{l}^\nu-C^{-2}u^\nu u^\lambda\nabla_\lambda u^\mu \qquad (\Omega^{\mu\nu}=-\Omega^{\nu\mu},\ \Omega^{\mu\nu}u_\mu=0) \tag{34}$$

i.e.

$$L=L'(q_l,u^\mu,\underset{i}{l}^\mu,\nabla_\mu q_l,\Omega^{\mu\nu},Q_a,\nabla_\mu Q_a,g_{\mu\nu}).$$

The tensor $\Omega^{\mu\nu}$ is the covariant generalisation of the angular velocity tensor for a space–time with torsion. We can use the above variational relations and, in this case, in addition take account of the fact that

$$\frac{\delta L}{\delta u^\mu}=\frac{\partial L'}{\partial u^\mu}+\frac{\partial L'}{\partial\Omega^{\alpha\beta}}\left[\underset{i}{l}^\alpha\nabla_\mu\underset{i}{l}^\beta-\frac{1}{c^2}(\delta^\beta_\mu u^\rho\nabla_\rho u^\alpha+u^\beta\nabla_\mu u^\alpha)\right]+\frac{1}{c^2}\overset{*}{\nabla}_\alpha\left(\frac{\partial L'}{\partial\Omega^{\mu\nu}}u^\nu u^\alpha\right),$$

$$\frac{dL}{\delta l^\mu}=\frac{\partial L'}{\partial \underset{i}{l}^\mu}+\frac{\partial L'}{\partial\Omega^{\mu\nu}}u^\rho\nabla_\rho\underset{i}{l}^\nu+\overset{*}{\nabla}_\rho\left(\frac{\partial L'}{\partial\Omega^{\mu\nu}}\underset{i}{l}^\nu u^\rho\right).$$

However, the transformation of variational relations for the concrete Lagrangians to the corresponding form when the derivatives $\nabla_\rho u^\mu$ and $\nabla_\rho \underset{i}{l}^\mu$ enter through the tensor $\Omega^{\mu\nu}$ is too cumbersome. In connection with this problem, it is more rational to develop a variational formalism for the case of an action integral in the form†

$$I=\frac{1}{c}\int_{(\Omega_4)}\mathscr{L}(A_B,\nabla_\mu A_B,g_{\mu\nu})\sqrt{-g}\,\mathrm{d}^4x, \qquad A_B=\{q_l,u^\mu,\Omega^{\mu\nu},Q_a\}. \tag{35}$$

To obtain the equations of translational motion we will determine the variation $\delta_F\Omega^{\mu\nu}$ corresponding to a variation (6) in displacement of the medium. Using (10), (12), (34) and the relation

$$\nabla_{[\mu}\nabla_{\nu]}A_B=\tfrac{1}{2}R^{\alpha}{}_{\cdot\beta\mu\nu}A_B|^{\beta}_{\cdot\alpha}-S_{\mu\nu}{}^{\cdot\cdot\alpha}\nabla_\alpha A_B,$$

† The dependence of $\mathscr{L}$ on the vectors $\underset{i}{l}^\mu$ besides the tensor $\Omega^{\mu\nu}$, if they are present, is easy to account for by means of the above-mentioned considerations for $L$.

we obtain

$$\nabla_F \Omega^{\mu\nu} = -\xi^\lambda (\nabla_\lambda \Omega^{\mu\nu} + n^\mu_\alpha n^\nu_\beta R^{\alpha\beta}_{\cdot\cdot\lambda\rho} u^\rho) + c^{-2} u^\rho (u_\lambda \Omega^{\mu\nu} + 2u^\alpha \nabla_\alpha u^{[\nu} n^{\mu]}_\lambda + 2u^{[\nu} \Omega^{\mu]}_{\cdot\lambda}) \xi^\lambda_{\cdot\rho}. \tag{36}$$

By virtue of (36) the equation of translational motion should be obtained in the form

$$(\overset{*}{\nabla}_\lambda \delta^\alpha_\nu - 2S_{\nu\lambda}^{\cdot\cdot\alpha}) \left[ \frac{\delta\mathscr{L}}{\delta u^\mu} n^\mu_\alpha u^\lambda - \frac{\delta\mathscr{L}}{\delta q_l} q_l n^\lambda_\alpha + \frac{1}{c^2} \frac{\delta\mathscr{L}}{\delta\Omega^{\mu\rho}} u^\lambda (u_\alpha \Omega^{\mu\rho} + 2u^\beta \nabla_\beta u^\rho n^\mu_\alpha + 2u^\rho \Omega^\mu_{\cdot\alpha}) \right] + \frac{\delta\mathscr{L}}{\delta u^\mu} \nabla_\nu u^\mu + \frac{\delta\mathscr{L}}{\delta q_l} \nabla_\nu q_l + \frac{\delta\mathscr{L}}{\delta\Omega^{\mu\lambda}} (\nabla_\nu \Omega^{\mu\lambda} + n^\mu_\alpha n^\lambda_\beta R^{\alpha\beta}_{\cdot\cdot\nu\rho} u^\rho) = 0. \tag{37}$$

Due to (14) and (34) the variation $\delta_r \Omega^{\mu\nu}$ corresponding to the variation in the triad $\underset{i}{l}^\mu$ has the form

$$\delta_r \Omega^{\mu\nu} = 2\varepsilon^{\alpha\beta} (\Omega^{[\mu}_{\cdot\alpha} + c^{-2} u^\alpha u^\rho \nabla_\rho u^{[\mu}) n^{\nu]}_\beta + n^\mu_\alpha n^\nu_\beta u^\rho \nabla_\rho \varepsilon^{\alpha\beta}. \tag{38}$$

Taking (38) into account, we find the equations of rotational motion in the form

$$2 \frac{\delta\mathscr{L}}{\delta\Omega^{\mu\nu}} \left( \Omega^\mu_{\cdot[\alpha} + \frac{1}{c^2} u^\rho \nabla_\rho u^\mu u_{[\alpha} \right) n^\nu_{\beta]} - \overset{*}{\nabla}_\rho \left( \frac{\delta\mathscr{L}}{\delta\Omega^{\mu\nu}} n^\mu_\alpha n^\nu_\beta u^\rho \right) = 0. \tag{39}$$

Noether's relations, obtained from the requirement of invariance of the Lagrangian $\mathscr{L}$, have the usual form (27) and (28) with a canonical energy–momentum tensor given by

$$t_\beta^{\cdot\alpha} = \mathscr{L} \delta^\alpha_\beta - \frac{\partial\mathscr{L}}{\partial(\nabla_\alpha A_B)} \nabla_\beta A_B - \frac{\delta\mathscr{L}}{\delta q_l} q_l n^\alpha_\beta + \frac{\delta\mathscr{L}}{\delta u^\mu} u^\alpha n^\mu_\beta + \frac{1}{c^2} \frac{\delta\mathscr{L}}{\delta\Omega^{\mu\nu}} u^\alpha (u_\beta \Omega^{\mu\nu} + 2u^\rho \nabla_\rho u^\nu n^\mu_\beta + 2u^\nu \Omega^\mu_{\cdot\beta}) \tag{40}$$

and the tensors $\sigma^{\lambda\nu\mu}$ and $\tau^{\alpha\beta}$ are defined in the following way:

$$\sigma^{\lambda\nu\mu} = 2 \frac{\partial\mathscr{L}}{\partial(\nabla_\mu A_B)} A_B|^{\lambda\nu} + 2 \frac{\delta\mathscr{L}}{\delta\Omega^{\alpha\beta}} n^{\alpha\lambda} n^{\beta\nu} u^\mu, \tag{41}$$

$$\tau^{\alpha\beta} = \mathscr{L} g^{\alpha\beta} + 2 \frac{\partial\mathscr{L}}{\partial g_{\alpha\beta}} - \frac{\delta\mathscr{L}}{\delta q_l} q_l n^{\alpha\beta} + \frac{1}{c^2} \frac{\delta\mathscr{L}}{\delta u^\lambda} u^\lambda u^\alpha u^\beta + \frac{1}{c^2} \frac{\delta\mathscr{L}}{\delta\Omega^{\mu\nu}} (\Omega^{\mu\nu} u^{(\alpha} + 2u^\rho \nabla_\rho u^\nu n^{\mu(\alpha} + 2u^\nu \Omega^{\mu(\alpha}) u^{\beta)}. \tag{42}$$

In order to define the metric energy–momentum tensor, we use relations (30), (31) and (34) and find the variation:

$$\delta_g \Omega^{\mu\nu} = [(2c^2)^{-1} \Omega^{\mu\nu} u^\alpha u^\beta + (g^{\alpha[\mu} - c^{-2} u^\alpha u^{[\mu}) \Omega^{\nu]\beta} + c^{-2} u^\alpha u^\rho \nabla_\rho u^{[\nu} n^{\mu]\beta}] \delta g_{\alpha\beta} + u^\beta n^{\alpha[\nu} n^{\mu]\lambda} \nabla_\lambda (\delta g_{\alpha\beta}).$$

Consequently, proceeding as usual, we obtain the metric energy–momentum tensor as

$$\theta^{\alpha\beta} = \mathscr{L}g^{\alpha\beta} + 2\frac{\partial\mathscr{L}}{\partial g_{\alpha\beta}} - \frac{\delta\mathscr{L}}{\delta q_l} q_l n^{\alpha\beta} + \frac{1}{c^2}\frac{\delta\mathscr{L}}{\delta u^\mu} u^\mu u^\alpha u^\beta$$

$$+ \frac{1}{c^2}\frac{\delta\mathscr{L}}{\delta\Omega^{\mu\nu}}[\Omega^{\mu\nu}u^\alpha u^\beta - 2(c^2 g^{\mu(\alpha} - u^\mu u^{(\alpha})\Omega^{\beta)\nu} + 2u^\rho\nabla_\rho u^\nu n^{\mu(\alpha}u^{\beta)}]$$

$$-\tfrac{1}{2}\overset{*}{\nabla}_\lambda(\sigma^{\lambda(\alpha\beta)} + \sigma^{(\alpha\beta)\lambda} - \sigma^{(\alpha|\lambda|\beta)}). \tag{43}$$

By virtue of (28) and (42) the connection between the canonical (40) and metric (43) energy–momentum tensors has the usual form (33), where the spin angular momentum tensor $\sigma^{[\mu\nu]\lambda}$ is defined according to (41). The metric energy–momentum tensors (32) and (43) satisfy the equations

$$\overset{*}{\nabla}_\mu\theta^{\mu\nu} = 0.$$

## 5. Some properties of the developed formalism

The above-developed variational formalism provides a unified description of conservative media with internal degrees of freedom and physical fields in a space–time with curvature and torsion. Sometimes, the function $L(\mathscr{L})$ can be represented as the sum of a Lagrangian for free fields $L_f(\mathscr{L}_f)$ and a material Lagrangian (with account of interactions) $L_m(\mathscr{L}_m)$

$$L = L_f(Q_a, \nabla_\mu Q_a, g_{\mu\nu}) + L_m(A_B, \nabla_\mu A_B, g_{\mu\nu}) \tag{44}$$

(similarly $\mathscr{L} = \mathscr{L}_f + \mathscr{L}_m$). Then the relations for the medium can be expressed through the function $L_m(\mathscr{L}_m)$. In accordance with (44), the total energy–momentum tensors are divided into two parts: the field one which has the usual form and the energy–momentum tensor of the medium. Therefore, in accordance with (25), the energy–momentum tensor of the medium may be represented in the form

$$\overset{m}{t}{}_\nu^{\cdot\mu} = P_\nu u^\mu - N n_\nu^\mu - (\partial L_m/\partial(\nabla_\mu A_B))\nabla_\nu A_B, \tag{45}$$

where the density of generalised momentum of the medium is given by

$$P_\nu = \frac{\delta L_m}{\delta u^\nu} + \frac{1}{c^2}\left(\frac{\delta L_m}{\delta u^\alpha}u^\alpha - L_m\right)u_\nu + \frac{1}{c^2}\frac{\delta\mathscr{L}_m}{\delta \underset{i}{l}{}^\alpha}u^\alpha \underset{i}{l}{}_\nu$$

and

$$N = (\delta L_m/(\delta q_l)q_l - L_m.$$

Using the action integral (35) in accordance with (40) the canonical energy–momentum tensor of the medium may be written in a form similar to (45) where this density of generalised momentum of the medium is

$$P_\nu = \frac{\delta\mathscr{L}_m}{\delta u^\nu} + \frac{1}{C^2}\left(\frac{\delta\mathscr{L}_m}{\delta u^\alpha}u^\alpha - \mathscr{L}_m\right)u_\nu + \frac{1}{C^2}\frac{\delta\mathscr{L}_m}{\delta\Omega^{\alpha\beta}}(u_\nu\Omega^{\alpha\beta} + 2u^\beta\Omega^{\alpha}_{\cdot\nu} + 2u^\rho\nabla_\rho u^\beta n_\nu^\alpha).$$

Due to the equations of translational motion (18), the energy-momentum tensor (45) satisfies the equation

$$(\overset{*}{\nabla}_\beta \delta^\alpha_\rho - 2S_{\rho\beta}^{\cdot\cdot\alpha})\overset{\mathrm{m}}{t}{}_\alpha^{\cdot\beta} = (\delta L_{\mathrm{m}}/\delta Q_a)\nabla_\rho Q_a - \tfrac{1}{2}\overset{\mathrm{m}}{\sigma}{}^{\alpha\beta\mu}R_{\alpha\beta\rho\mu}, \tag{46}$$

where $\overset{\mathrm{m}}{\sigma}{}^{[\alpha\beta]\mu} = 2(\partial L_{\mathrm{m}}/\partial(\nabla_\mu A_B))A_B|^{[\alpha\beta]}$ is the spin angular momentum tensor of the medium. Using the condition of invariance of $L_{\mathrm{m}}$ written in the form

$$2\frac{\partial L_{\mathrm{m}}}{\delta g_{\mu\nu}} - \frac{\delta L_{\mathrm{m}}}{\delta A_B}A_B|^{\nu\mu} + \frac{\partial L_{\mathrm{m}}}{\partial(\nabla_\nu A_B)}g^{\mu\lambda}\nabla_\lambda A_B - \overset{*}{\nabla}_\lambda\left(\frac{\partial L_{\mathrm{m}}}{\partial(\nabla_\lambda A_B)}A_B|^{\nu\mu}\right) = 0$$

and also the equations of rotational motion (19), one can obtain the relation

$$\overset{*}{\nabla}_\alpha \overset{\mathrm{m}}{\sigma}{}^{[\beta\lambda]\alpha} = 2\overset{\mathrm{m}}{t}{}^{[\beta\lambda]} - 2(\delta L_{\mathrm{m}}/\delta Q_a)Q_a|^{[\beta\lambda]}. \tag{47}$$

Note that relations of the form (46) and (47) are fulfilled for the Lagrangian $\mathscr{L}_{\mathrm{m}}$. Equations (46) and (47) represent another form of the equations of translational and rotational motions of the medium.

It is easy to show that the formalism developed is in full agreement with the relativistic dynamics of oriented particles. To this end, let us consider the case when the Lagrangian does not depend on the derivatives, i.e.

$$\mathscr{L}_{\mathrm{m}} = \mathscr{L}_{\mathrm{m}}(q_l, u^\mu, \underset{i}{l}{}^\mu, \Omega^{\mu\nu}, Q_a, g_{\mu\nu})$$

and $\mathscr{L}_{\mathrm{m}}$ depends linearly and homogeneously on the densities $q_l$. For such a Lagrangian $N = 0$ and the variational relations simplify to a great extent:

$$\overset{\mathrm{m}}{t}{}_\nu^{\cdot\mu} = P_\nu u^\mu,$$

$$\overset{\mathrm{m}}{\sigma}{}^{\mu\nu\lambda} = M^{\mu\nu}u^\lambda = 2(\partial\mathscr{L}_{\mathrm{m}}/\partial\Omega^{\alpha\beta})n^{\alpha\mu}n^{\beta\nu}u^\lambda,$$

$$\begin{aligned} P_\nu = {}& \frac{\partial\mathscr{L}_{\mathrm{m}}}{\partial u^\nu} + \frac{1}{c^2}\left(\frac{\partial\mathscr{L}_{\mathrm{m}}}{\partial u^\alpha}u^\alpha - \mathscr{L}_{\mathrm{m}}\right)u_\nu + \frac{1}{c^2}\frac{\partial\mathscr{L}_{\mathrm{m}}}{\partial \underset{i}{l}{}^\alpha}u^\alpha \underset{i}{l}{}_\nu \\ & + \frac{1}{c^2}\frac{\partial\mathscr{L}_{\mathrm{m}}}{\partial\Omega^{\alpha\beta}}(u_\nu\Omega^{\alpha\beta} + 2u^\beta\Omega^{\alpha}_{\cdot\nu} + 2u^\rho\nabla_\rho u^\beta n^\alpha_\nu), \end{aligned} \tag{48}$$

$$(\overset{*}{\nabla}_\beta\delta^\alpha_\rho - 2S_{\rho\beta}^{\cdot\cdot\alpha})(P_\alpha u^\beta) = (\partial\mathscr{L}_{\mathrm{m}}/\partial Q_a)\nabla_\rho Q_a - \tfrac{1}{2}M^{\alpha\beta}R_{\alpha\beta\rho\lambda}u^\lambda,$$

$$\overset{*}{\nabla}_\rho(M_{\mu\nu}u^\rho) = 2M_{\alpha[\nu}(\Omega^{\alpha}_{\cdot\mu]} + C^{-2}u_{\mu]}u^\rho\nabla_\rho u^\alpha) + 2(\partial\mathscr{L}_{\mathrm{m}}/\partial \underset{i}{l}{}^\alpha)\underset{i}{l}{}_{[\mu}n^\alpha_{\nu]}.$$

The equations (48) are similar to the corresponding equations governing the relativistic dynamics of a particle†. As an example of the application of the proposed formalism let us consider a medium of the type of Cosserat's media (Salié 1973). Neglecting the dependence of the potential energy of elastic compression of the medium $\Pi$ (per unit of mass) on rotational degrees of freedom, we write the Lagrangian in the form

$$\mathscr{L}_{\mathrm{m}} = -\mu c^2 - \mu\Pi(\mu) + \tfrac{1}{4}\mu\mathscr{I}\Omega^{\alpha\beta}\Omega_{\alpha\beta} \qquad (\mathscr{I} = \text{constant}) \tag{49}$$

where $\mu$ is the invariant mass density. As a consequence of the formula $\mathrm{d}\Pi = p\,\mathrm{d}\mu/\mu^2$

† Lagrange's relativistic function of a particle can be obtained from $\mathscr{L}_{\mathrm{m}}$ by substituting for the invariant densities $q_l$ the corresponding constant charges of the particle.

were $p$ is the invariant pressure, we can write the variational relations (48) corresponding to the Lagrangian (49) as

$$
\begin{aligned}
&P_\nu u_\nu = \rho u_\nu + c^{-2} M_{\nu\beta} u^\rho \nabla_\rho u^\beta, \\
&\overset{\mathrm{m}}{\sigma}{}^{\mu\nu\lambda} = M^{\mu\nu} u^\lambda, \qquad M^{\mu\nu} = \mu \mathscr{I} \Omega^{\mu\nu}, \\
&\overset{\mathrm{m}}{t}{}^{\alpha\beta} = (\rho + p/c^2) u^\alpha u^\beta + p g^{\alpha\beta} + c^{-2} M^{\alpha}_{\cdot\rho} u^\lambda \nabla_\lambda u^\rho u^\beta, \\
&(\overset{*}{\nabla}_\beta \delta^\alpha_\rho - 2S_{\rho\beta}^{\cdot\cdot\alpha})[(\rho + p/c^2) u_\alpha u^\beta + p\delta^\beta_\alpha + c^{-2} M_{\alpha\rho} u^\lambda \nabla_\lambda u^\rho u^\beta] = -\tfrac{1}{2} M^{\alpha\beta} u^\mu R_{\alpha\beta\rho\mu}, \\
&u^\rho \nabla_\rho \Omega_{\alpha\beta} = (2/c^2) \Omega_{\mu[\beta} u_{\alpha]} u^\lambda \nabla_\lambda u^\mu,
\end{aligned}
\tag{50}
$$

where the total mass density

$$\rho = -P_\nu u^\nu / c^2 = \mu[1 + c^{-2}(\Pi + \tfrac{1}{4}\mathscr{I}\Omega^{\alpha\beta}\Omega_{\alpha\beta})].$$

For $\Pi \to 0$ and $p \to 0$ the equations (50) give the continuum form corresponding to the equations of rotating particles which are a generalisation of the known Papapetrou (1951) equations for the case of a space–time with torsion (Ponomariev 1973, Trautman 1973, Minkevich and Sokolski 1975).

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